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# Pathway Fractional Integral Formulas Associated with the Incomplete ℵ-Functions

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## Abstract
In the present work, we establish two pathway fractional integral formulas associated with the Incomplete ℵ-functions. Further, we develop some special cases involving various simpler and useful special functions are given to show the importance and utilizations of our main findings. After that we indicate some known results, Bansal and Choi [1] and Nair [2] reduced by our main findings.

## Keywords
Pathway fractional integral operator, Incomplete Aleph-function, Mellin-Barnes type contour, Riemann-Liouville fractional integral operators

## 2010 Mathematics Subject Classification
Primary 26A33, 33B20; Secondary 33C60, 33E20, 44A40

## 1. Introduction/Definitions
Many research efforts have been devoted to generalize Fractional calculus (FC) and special function. FC become a significant instrument for the modeling, analysis and plays a crucial role in different fields. Fractional calculus has recently attracted considerable attention and importance during past four decades. It is prescribed the definitions of differentiation and integration to arbitrary order. FC is a fascinating branch of applied sciences and it also represents a powerful tool to examine a Myriad of problems from different fields like Mathematical modeling, Probability theory, Physics, Control theory and various other problem related to differential and integral equations, partial differential equations associated with special functions [3-6].

During four decades, several researchers have given some interesting and useful generalization of many of the familiar special function such as various type of Mittag-Leffler functions, generalized Bessel function, type of hypergeometric function, incomplete Gamma function, incomplete Euler's-Beta function, incomplete H-function, incomplete I-function, incomplete Aleph function, etc. [7–11].

Recently, Bansal et al. [12] explored and investigated the incomplete Aleph function which is generalization of incomplete I-function, incomplete H-function, the ℵ-function and I-function and many other special cases belongs to Fox's H-function.

The main object of this paper is to establish a set of new and interesting pathway fractional integral formulae involving incomplete aleph function and produce some known and unknown results of our main finding.
We recall here classical definition of incomplete Gamma functions $\Gamma(\nu, z)$ and $\gamma(\nu, z)$ defined as follows:

$$\gamma(\nu, z) = \int_0^x e^{-t} t^{\nu-1} dt \qquad (\Re(\nu) > 0\, ; z \geq 0) \qquad (1.1)$$

And

$$\Gamma(\nu,z)=\int_x^\infty e^{-t}t^{\nu-1}dt \qquad (\Re(\nu)>0\,; z\geq 0 \; if \;\; z=0) \qquad (1.2)$$

The incomplete Gamma functions $\Gamma(\nu,z)$ and $\gamma(\nu,z)$ are satisfying decomposition formula:

$$\gamma(\nu,z)+\Gamma(\nu,z)=\Gamma(\nu) \qquad (\Re(\boldsymbol{\nu})>\mathbf{0}) \qquad (1.3)$$

Very recently Bansal et. al [12] introduced and investigate the incomplete Aleph function ${}^{(\Gamma)}\aleph^{m,n}_{p_i,q_i,\rho_i,r}(z)$ and $(\gamma)_{\aleph}{}^{m,n}_{p_i,q_i,\rho_i,r}(z)$ which are defined by Mellin-Barnes type contour integral representations as follows:

$$(\gamma)\aleph^{m,n}_{p_i,q_i,\rho_i,r}(z)={}^{(\gamma)}(\aleph)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\left|\begin{matrix}(a_1,\varkappa_1,y),(a_j,\varkappa_j)_{2,n},[\rho_j(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\(b_j,\mathcal{B}_j)_{1,m},[\rho_j(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\theta(\xi,y)(z)^{-\xi}d\xi, \qquad (1.4)$$

*Where* $z\neq 0$, and

$$\theta(\xi,y)=\frac{\Gamma(1-a_1-\varkappa_1\xi,y)\prod_{j=1}^{m}\Gamma(b_j+\mathcal{B}_j\xi)\prod_{j=2}^{n}\Gamma(1-a_j-\varkappa_j\xi)}{\sum_{i=1}^{r}\rho_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\mathcal{B}_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\varkappa_{ji}\xi)\right]} \qquad (1.5)$$

and

$$(\gamma)_{\aleph}{}^{m,n}_{p_i,q_i,\rho_i,r}(z)={}^{(\gamma)}(\aleph)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\left|\begin{matrix}(a_1,\varkappa_1,y),(a_j,\varkappa_j)_{2,n},[\rho_j(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\(b_j,\mathcal{B}_j)_{1,m},[\rho_j(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_{\mathcal{L}}\Theta(\xi,y)(z)^{-\xi}d\xi, \qquad (1.6)$$

*Where* $z\neq 0$, and

$$\Theta(\xi,y)=\frac{\gamma(1-a_1-\varkappa_1\xi,y)\prod_{j=1}^{m}\Gamma(b_j+\mathcal{B}_j\xi)\prod_{j=2}^{n}\Gamma(1-a_j-\varkappa_j\xi)}{\sum_{i=1}^{r}\rho_i\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\mathcal{B}_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\varkappa_{ji}\xi)\right]} \qquad (1.7)$$

The incomplete Aleph function given in (1.5) and (1.7) exist for all y>=0 under the same contour and the same set of conditions as stated in [12]. A complete detail of (1.5) and (1.7) can be found in [12].Some important particular cases of incomplete aleph function are listed below:

(i) On taking y = 0, then (1.7) reduces to the ℵ−function which was given by *Sũdland* [13, 14] as follows:

$${}^{(\gamma)}(\aleph)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\left|\begin{matrix}(a_1,\varkappa_1,0),(a_j,\varkappa_j)_{2,n},[\rho_j(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\(b_j,\mathcal{B}_j)_{1,m},[\rho_j(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right]=\aleph^{m,n}_{p_i,q_i,\rho_i,r}\left[z\left|\begin{matrix}(a_j,\varkappa_j)_{1,n},[\rho_j(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\(b_j,B_j)_{1,m},[\rho_j(b_{ji},B_{ji})]_{m+1,q_i}\end{matrix}\right.\right] \qquad (1.8)$$

(ii) On setting $\rho_i=1$, then equations (1.5) and (1.7) reduces to the incomplete *I*- functions introduced by Bansal and Kumar [15] as given below:

$${}^{(\gamma)}(\aleph)^{m,n}_{p_i,q_i,1,r}\left[z\left|\begin{matrix}(a_1,\varkappa_1,0),(a_j,\varkappa_j)_{2,n},[(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\(b_j,\mathcal{B}_j)_{1,m},[(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right]$$

$$={}^{(\gamma)}(I)^{m,n}_{p_i,q_i,r}\left[z\left|\begin{matrix}(a_1,\varkappa_1,y)_{1,n},(a_j,\varkappa)_{2,n},[1(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\(b_j,\mathcal{B}_j)_{1,m},[(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right] \qquad (1.9)$$

And

$${}^{(\gamma)}(\aleph)^{m,n}_{p_i,q_i,,r}\left[z\left|\begin{matrix}(a_1,\varkappa_1,0),(a_j,\varkappa_j)_{2,n},[(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\(b_j,\mathcal{B}_j)_{1,m},[(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right]$$

$$={}^{(\gamma)}(I)^{m,n}_{p_i,q_i,\rho_i,r}\left[z\left|\begin{matrix}(a_1,\varkappa_1,y)_{1,n},(a_j,\varkappa_j)_{2,n},[(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\(b_j,\mathcal{B}_j)_{1,m},[(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right] \qquad (1.10)$$

(iii) For $\rho_i = 1$ *and* $y = 0$ in equation (1.6), then it's reduces to the I-functions given by Saxena [16] as follows:

$$^{(\gamma)}(\aleph)^{m,n}_{p_i,q_i,1,r}\left[z\left|\begin{matrix}(a_1,\varkappa_1,0),(a_j,\varkappa_j)_{2,n},[(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\(b_j,\mathcal{B}_j)_{1,m},[(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right]=I^{m,n}_{p_i,q_i,r}\left[z\left|\begin{matrix}(a_1,\varkappa_1)_{1,n},(a_{ji},\varkappa_{ji})_{n+1,p_1}\\(b_j,\mathcal{B}_j)_{1,m},(b_{ji},\mathcal{B}_{ji})_{m+1,q_1}\end{matrix}\right.\right] \quad (1.11)$$

(iv) Further setting $\rho_i = 1$ *and* $r = 1$ in equations (1.4) and (1.6) reduces into the incomplete H-function introduced by Srivastava et al.[17] and (see also [1]) as follows:

$$^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,1,1}\left[z\left|\begin{matrix}(a_1,\varkappa_1,y),(a_j,\varkappa_j)_{2,n},(a_{ji},\varkappa_{ji})_{n+1,i}\\(b_j,\mathcal{B}_j)_{1,m},[1(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right]=\Gamma^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,\varkappa_1,y)\ \ ,(a_j,\varkappa_j)_{2,p}\\(b_j,\mathcal{B}_j)_{1,q}\end{matrix}\right.\right] \quad (1.12)$$

$$^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,1,1}\left[z\left|\begin{matrix}(a_1,\varkappa_1,y),(a_j,\varkappa_j)_{2,n},[1(a_{ji},\varkappa_{ji})]_{n+1,p_{ii}}\\(b_j,\mathcal{B}_j)_{1,m},[1(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right.\right]=\gamma^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_1,\varkappa_1,y),(a_j,\varkappa_j)_{2,p}\\(b_j,\mathcal{B}_j)_{1,q}\end{matrix}\right.\right] \quad (1.13)$$

A complete detail of Incomplete H-function can be found in the article [17].

(v) Again setting y = 0 in (1.12), incomplete ℵ-function deduce into the Fox's H-function [18] as follows:

$$^{(\Gamma)}(\aleph)^{m,n}_{p_i,q_i,1,1}\left[z\left|\begin{matrix}(a_1,\varkappa_1,0),(a_j,\varkappa_j)_{2,n},[1(a_{ji},\varkappa_{ji})]_{n+1,p_1}\\(b_j,\mathcal{B}_j)_{1,m},[1(b_{ji},\mathcal{B}_{ji})]_{m+1,q_1}\end{matrix}\right.\right]=H^{m,n}_{p,q}\left[z\left|\begin{matrix}(a_j,\varkappa_j)_{1,p}\\(b_j,\mathcal{B}_j)_{1,q}\end{matrix}\right.\right] \quad (1.14)$$

Let $\Phi \in \mathcal{L}(a,b)$ the set of Lebesgue measurable functions defined on (a, b). Then the pathway fractional integral operator $P^{(\lambda,\mu,a)}_{0^+}\Phi$ with a pathway parameter $\mu < 1$ is defined as follows [1]:

$$P^{(\lambda,\mu,a)}_{0^+}\Phi = \int_0^{\frac{x}{a(1-\mu)}}\left[1-\frac{a(1-\mu)u}{y}\right]^{\frac{\lambda}{1-\mu}}\Phi(u)du, \quad (1.15)$$

$(\Re(\lambda) > 0, a\epsilon\mathbb{R}^+, \Phi \in \mathcal{L}(0,\infty))$

It is reduce to well-known definition of Riemann-Liouville fractional integral operators $I^{\lambda}_{0^+}$[1] as follows:

$$\left(P^{(\lambda-1,0,1)}_{0^+}\Phi\right)(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\lambda}}dt = \Gamma\lambda\left(I^{\lambda}_{0^+}\Phi\right)(x) \quad (1.16)$$

We also recall a known result (see [2, Eq. (12)])

$$\left(P^{(\lambda,\mu,a)}_{0^+}\right)(x^{\nu-1}) = \frac{x^{\lambda+\nu}}{[a(1-\mu)]^{\nu}}\frac{\Gamma(\nu)\Gamma\left(1+\frac{\lambda}{1-\mu}\right)}{\Gamma\left(1+\nu+\frac{\lambda}{1-\mu}\right)} \quad (1.17)$$

$$(\mu < 1, \Re(\lambda) > 0, a\epsilon\mathbb{R}\hat{}+, \Re(\nu) > 0, \Re(1+\lambda/(1-\mu)) > 0)$$

## 2. Pathway Fractional Integral Formulas of the Incomplete ℵ -Functions

We derived pathway fractional integral formulas of the incomplete ℵ-Functions (1.4) and (1.6).

**Theorem 1** *let* $a \in \mathbb{R}^+, \mu < 1, \beta \in \mathbb{R}^+, \Re(\alpha) > 0\ \Re\left(1+\frac{\lambda}{1-\mu}\right) > 0, y \geq 0$ *and* $k \in \mathbb{R}$. Also let $a_j \in \mathbb{C}, \varkappa_j \in \mathbb{R}^+\ (j = 1, \ldots. p)$ *and* $b_j \in \mathbb{C}, \mathcal{B}_j \in \mathbb{R}^+\ (j = 1, \ldots. q)$ *be the same as in* (1.4). *Then*

$$P^{(\lambda,\mu,a)}_{0^+}\left\{x^{\alpha-1\,(\Gamma)}\aleph^{m,n}_{p_iq_{i,\rho_{i,,}r}}(kx^{\beta}\right\} = x^{\lambda+\alpha}\frac{\Gamma\left(\left(1+\frac{\lambda}{1-\mu}\right)\right)}{[a(1-\mu)]^{\alpha}}$$

$$^{(\Gamma)}\aleph^{m,n+1}_{p_i+1,q_{i,\rho_{i,,}r}}\left[\frac{kx^{\beta}}{[a(1-\mu)]^{\beta}}\middle| \begin{matrix}(a_1,\varkappa_1,y),(1-\alpha,\beta),(a_j,\varkappa_j)_{2,n},[\rho_j(a_{j1},\varkappa_{j1})]_{n+1,p_i}\\ \left(-\alpha-\frac{\lambda}{1-\mu},\beta\right),(b_j,\mathcal{B}_j)_{1,m},[\rho_j(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right] \tag{2.1}$$

It is assumed that each member of (2.1) exists.

***Proof:*** Suppose $\Delta$ be the left hand side of (2.1). Using (1.15) and (1.4), we have

$$\Delta= P^{(\lambda,\mu,a)}_{0^+}\left\{x^{\alpha-1}\frac{1}{2\pi i}\int_{\mathcal{L}}\Theta(\xi,y)(z)^{-\xi}d\xi,\frac{1}{2\pi i}\int_{\mathcal{L}}\theta(\xi,y)(kx^{\beta})^{-\xi}d\xi\right\} \tag{2.2}$$

Where $\theta(\xi, y)$ is given in (1.5). Then, in the double integral of the right member of (2.2), we interchange the order of the integrals, which can be verified under the stated conditions, we get

$$\Delta=\frac{1}{2\pi i}\int_{\mathcal{L}}\theta(\xi,y)k^{-\xi}\ P^{(\lambda,\mu,a)}_{0^+}\{x^{\alpha-\beta\xi-1}\}d\xi \tag{2.3}$$

Using (1.17) to evaluate the pathway fractional integral in (2.3), we get

$$\Delta= x^{\lambda+\alpha}\frac{\Gamma\left(1+\frac{\lambda}{1-\mu}\right)}{[a(1-\mu)]^{\alpha}}\frac{1}{2\pi i}\int_{\mathcal{L}}\theta(\xi,y)(k)^{-\xi}\left(\frac{x}{a(1-\mu)}\right)^{-\beta\xi}\frac{\Gamma(\alpha-\beta\xi)}{\Gamma\left(\alpha-\beta\xi+\frac{\lambda}{1-\mu}+1\right)}d\xi$$

Finally, with the help of (1.4) and (1.5), interpreting the right member of the last identity, we obtain the desired result (2.1).

**Theorem 2** *let* $a\in\mathbb{R}^+,\mu<1,\beta\in\mathbb{R}^+,\Re(\alpha)>0\ \Re\left(1+\frac{\lambda}{1-\mu}\right)>0,y\geq 0$ *and* $k\in\mathbb{R}$. Also let $a_j\in\mathbb{C},\varkappa_j\in\mathbb{R}^+\ (j=1,\dots.p)$ *and* $b_j\in\mathbb{C},\mathcal{B}_j\in\mathbb{R}^+\ (j=1,\dots.q)$ *be the same as in* (1.6). *Then*

$$P^{(\lambda,\mu,a)}_{0^+}\left\{x^{\alpha-1}\,{}^{(\gamma)}\aleph^{m,n}_{p_iq_{i,\rho_{i,,}r}}(kx^{\beta}\right\}=x^{\lambda+\alpha}\frac{\Gamma\left(\left(1+\frac{\lambda}{1-\mu}\right)\right)}{[a(1-\mu)]^{\alpha}}$$

$$\times\,^{(\gamma)}\aleph^{m,n+1}_{p_i+1,q_{i+1,\rho_{i;}r}}\left[\frac{kx^{\beta}}{[a(1-\mu)]^{\beta}}\middle| \begin{matrix}(a_1,\varkappa_1,y),(1-\alpha,\beta),(a_j,\varkappa_j)_{2,n},[\rho_j(a_{j1},\varkappa_{j1})]_{n+1,p_i}\\ \left(-\lambda-\frac{\eta}{1-\alpha},\tau\right),(b_j,\mathcal{B}_j)_{1,m},[\rho_j(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right] \tag{2.4}$$

It is assumed that each member of (2.4) exists.

***Proof:*** The proof of (2.4) would be written as similar lines of Theorem 1. We drop the details.

## 3. Special Cases and Remarks

Several special cases can be obtained from the results (2.1) and (2.4), some of them are listed below in the form of Corollaries:

**Corollary 1** *let* $a\in\mathbb{R}^+,\mu<1,\beta\in\mathbb{R}^+,\Re(\alpha)>0\ \Re\left(1+\frac{\lambda}{1-\mu}\right)>0,y\geq 0$ *and* $k\in\mathbb{R}$. Also let $a_j\in\mathbb{C},\varkappa_j\in\mathbb{R}^+\ (j=1,\dots.p)$ *and* $b_j\in\mathbb{C},\mathcal{B}_j\in\mathbb{R}^+\ (j=1,\dots.q)$ *be the same as in* (1.4). *Then*

$$P^{(\lambda,\eta,a)}_{0^+}\left\{x^{\alpha-1}\aleph^{m,n}_{p_iq_{i,\rho_{i,,}r}}(kx^{\beta}\right\}=x^{\alpha+\lambda}\Gamma((1+\eta))$$

$$\times\,\aleph^{m,n+1}_{p_i+1,q_{i+1,\rho_{i,,}r}}\left[\frac{kx^{\beta}}{[a(1-\mu)]^{\beta}}\middle| \begin{matrix}(1-\alpha,\tau),(a_j,\varkappa_j)_{2,n},[\rho_j(a_{ji},\varkappa_{ji})]_{n+1,p_i}\\ \left(-\alpha-\frac{\lambda}{1-\mu},\beta\right),(b_j,\mathcal{B}_j)_{1,m},[\rho_j(b_{ji},\mathcal{B}_{ji})]_{m+1,q_i}\end{matrix}\right] \tag{3.1}$$

It is assumed that each member of (3.1) exists.

***Proof:*** In consideration of (1.8) and (2.1), we obtain the desired result (3.1).

**Corollary 2** *let* $a\in\mathbb{R}^+,\mu<1,\beta\in\mathbb{R}^+,\Re(\alpha)>0\ \Re\left(1+\frac{\lambda}{1-\mu}\right)>0,y\geq 0$ *and* $k\in\mathbb{R}$. Also let $a_j\in\mathbb{C},\varkappa_j\in\mathbb{R}^+\ (j=1,\dots.p)$ *and* $b_j\in\mathbb{C},\mathcal{B}_j\in\mathbb{R}^+\ (j=1,\dots.q)$ *be the same as in* (1.4) *and* (1.6). *Then*

$$P^{(\lambda,\mu\mu,a)}_{0^+}\left\{x^{\alpha-1}I^{m,n}_{p_iq_{i,\rho_{i,,}r}}(kx^{\beta})\right\}=x^{\alpha+\lambda}\frac{\Gamma\left(\left(1+\frac{\lambda}{1-\mu}\right)\right)}{[a(1-\mu)]^{\alpha}}$$

$$\times\ {}^{(\gamma)}I^{m,n+1}_{p_i+1,q_{i+1,\rho_{i,,}r}}\left[\frac{kx^{\beta}}{[a(1-\mu)]^{\beta}}\middle|\begin{matrix}(a_j,\varkappa_j,y),(1-\alpha,\beta),(a_j,\varkappa_j)_{2,n},(a_{j1},A_{ji})_{n+1,p_i}\\ \left(-\lambda-\frac{\eta}{1-\alpha},\tau\right),(b_j,B)_{1,m},[\rho_j(b_{j1},b_{j1})]_{m+1,q_i}\end{matrix}\right] \quad (3.2)$$

***Proof:*** With the help of (1.9) and (1.10), we can get the results here from those in Theorem 1 and 2

**Corollary 3** Let $A_1>0,\ |\arg z|<\frac{\pi}{2}A_1\ (i=1\ldots..r)\ and\ \Re(\Phi_1)+1<0$. Also let $a\in\mathbb{R}^+$ $\alpha<1,\tau\in\mathbb{R}^+,\Re(\lambda)>0\ \Re\left(1+\frac{\eta}{1-\alpha}\right)>0,x\geq 0\ and\ k\in\mathbb{R}$. be the same as in (1.6). Then

$$I^{(\eta)}_{0^+}\left\{z^{\lambda-1}\,{}^{(\Gamma)}\aleph^{m,n}_{p_iq_{i,\rho_{i,,}r}}(kz^{\tau})\right\}=\frac{\Gamma\left(\left(1+\frac{\eta}{1-\alpha}\right)\right)}{[a(1+\alpha)]^{\lambda}}$$

$$\times\ \aleph^{m,n+1}_{p_i+1,q_{i,\rho_{i,,}r}+1}\left[\frac{kz^{\tau}}{[a(1+\alpha)]^{\tau}}\middle|\begin{matrix}(1-\lambda,\tau),(a_j,A_j)_{1,n},[\rho_j(a_{j1},A_{j1})]_{n+1,p_i}\\ \left(-\lambda-\frac{\eta}{1-\alpha},\tau\right),(b_j,B)_{1,m},[\rho_j(b_{j1},b_{j1})]_{m+1,q_i}\end{matrix}\right] \quad (3.4)$$

***Proof:*** Further setting y = 0 in (3.3), the identity here follows

**Corollary 4** Let $A_1>0,\ |\arg z|<\frac{\pi}{2}A_1\ (i=1\ldots..r)\ and\ \Re(\Phi_1)+1<0$. Also let $a\in\mathbb{R}^+$ $\alpha<1,\tau\in\mathbb{R}^+,\Re(\lambda)>0\ \Re\left(1+\frac{\eta}{1-\alpha}\right)>0,x\geq 0\ and\ k\in\mathbb{R}$..

$$P^{(\eta,\alpha,a)}_{0^+}\left\{z^{\lambda-1}\,{}^{(\Gamma)}I^{m,n}_{p_iq_{i,\rho_{i,,}r}}(kz^{\tau}\right\}=z^{\eta+\lambda}\frac{\Gamma\left(\left(1+\frac{\eta}{1-\alpha}\right)\right)}{[a(1+\alpha)]^{\lambda}}$$

$${}^{(\Gamma)}I^{m,n+1}_{p_i+1,q_{i,r}+1}\left[\frac{kz^{\tau}}{[a(1+\alpha)]^{\tau}}\middle|\begin{matrix}(a_1,A_1,y),(1-\lambda,\tau),(a_j,A_j)_{2,n},[(a_{j1},A_{j1})]_{n+1,p_i}\\ \left(-\lambda-\frac{\eta}{1-\alpha},\tau\right),(b_j,B)_{1,m},[(b_{j1},b_{j1})]_{m+1,q_i}\end{matrix}\right] \quad (3.5)$$

*and*

$$P^{(\eta,\alpha,a)}_{0^+}\left\{z^{\lambda-1}\,{}^{(\Gamma)}I^{m,n}_{p_iq_{i,\rho_{i,,}r}}(kz^{\tau}\right\}=z^{\eta+\lambda}\frac{\Gamma\left(\left(1+\frac{\eta}{1-\alpha}\right)\right)}{[a(1+\alpha)]^{\lambda}}$$

$${}^{(\gamma)}I^{m,n+1}_{p_i+1,q_{i,r}+1}\left[\frac{kz^{\tau}}{[a(1+\alpha)]^{\tau}}\middle|\begin{matrix}(a_1,A_1,y),(1-\lambda,\tau),(a_j,A_j)_{2,n},[(a_{j1},A_{j1})]_{n+1,p_i}\\ \left(-\lambda-\frac{\eta}{1-\alpha},\tau\right),(b_j,B)_{1,m},[(b_{j1},b_{j1})]_{m+1,q_i}\end{matrix}\right] \quad (3.6)$$

***Proof***: In view of (1.16), we get the required result directly from the Theorem (1) and (2). By suitably specializing the parameters of incomplete ℵ-functions (1.4) and (1.6), several known results can be deduced from the main findings (2.1) and (2.4) in which few of them given below:

***Remark 1***: If, we reduce the incomplete ℵ-functions to the incomplete H-functions with the help of (1.12) and (1.13) in Theorem 1 and 2, we get the results obtained by Bansal and Choi [1]. Similarly, If we reduce the incomplete Aleph functions into the Fox's H-function with the help of (1.14) in Theorem 2, we get the identities recorded by Nair [2].

## 4. Conclusion

Fractional integral operators are very used in Physics, Mathematics and engineering applications. In this investigation, we established two pathway fractional integral formulae associated with incomplete ℵ−functions. Next we found some new and known results of the main findings. These results are very use full for further research and applications in the field of Engineering, Physics, and Mathematics.

### Statements and Declarations

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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