Zhongguo Kuangye Daxue Xuebao

2025 | Vol 30 | Issue 3 | Page 76-80
Journal Homepage: https://zkdx.ch/
DOI: 10.1654/zkdx.2025.30.3-08

Construction of a Fourth Order Iterative Method to Solve Non-Linear Equations

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Abstract
In this paper, we have developed the fourth order iterative method to find the simple roots of the non-linear equations that arise in engineering and scientific fields. Through theoretical derivatives and practical examples, we illustrate how to evaluate the simple roots of non-linear equations. The proposed scheme demonstrates efficient results as well as improves the computational performance. The study concluded by discussing potential applications and implications for future research in simple root finding methods.

Keywords
Iterative Methods, Order of Convergence, Non-Linear Equations

1. Introduction
One of the most important problems in engineering and sciences is solving the non-linear equations. The non-linear equation's solution H(q) = 0 has been one of the most investigated topics in applied mathematics, which produces a vast literature. To find the exact analytical solution of certain problems is very difficult or impossible. Iterative methods are techniques that make successive approximations to arrive at a more accurate solution. When a problem is too difficult to solve directly (analytically), we use a step-by-step process in which the method gradually gets closer to the correct answer over time. So, the main part is that the complexity of certain problems leads us to use approximation techniques instead of exact solution. The best illustration of iterative procedures is Newton's method [1], which is described as,

$$q_{n+1} = q_n - \frac{H(q_n)}{H'(qn)}, n = 0,1,2 \dots \dots \dots (1)$$

The Newton's method exhibits quadratic convergence. To increase Newton's method's convergence order from quadratic to cubic, numerous researchers [2-5] have contributed to the development of iterative methods. Some of the well-established cubically convergent methods include Halley's method [2], Euler's method [3], the super-Halley method [4], and the Weekaroon-Fernando method [5], among others. Out of these, all methods, except for the Weekaroon-Fernando method, involve second-order derivatives. From 1964 and till now, researchers [6-10] developed fourth-order methods to determine the roots of non-linear equations, including those proposed by Traub and Ostrowski [6], Chun and Ham [7], Cordero and Torregrosa [8], Singh and Bhalla [9], and Kanwar et al. [10]. Among these, Kanwar et al. introduced a method incorporating second-order derivatives, while the remaining approaches utilized first-order derivatives. Obtaining the second-order derivative can sometimes be challenging. Considering this, recent studies [11-13] have focused on developing methods that do not require second order derivatives. In this context, we propose a novel fourth-order method.

The remainder of the paper is structured as follows: In section (2), we introduce a fourth order scheme for solving non-linear equation and analyses their convergence. Section (3) focuses on evaluating the method's numerical performance in comparison to other current methods. Lastly, the conclusion is presented in Section (4).

2. The Proposed Method and its Convergence Analysis

Using Newton's approach as a first step, the following iterative scheme is developed. The expression that is iterative,

$$w_n = q_\eta - \frac{H(q_n)}{H'(q_n)},$$
$$q_{n+1} = q_n - \frac{H^2(q_n) - 2H^2(w_n) + H(w_n)H(q_n)}{H(q_n) - H'(q_n)}, n = 0,1,2,3, \ldots \ldots \ldots (2)$$

There are three fractional evaluations for each member of family (2) each iteration. The following results indicate that the family is optimal because of its four order of convergence.

Theorem: In an open interval I, let H: I ⊂ R → R be a real sufficiently differentiable function, and let α ∈ I be a simple root of H(q) = 0. By being its error equation, if e_n is close enough to α, the iterative family (2) converges to α with order of convergence four.

$$e_{n+1} = (5c_2^3 - c_2c_3)e_n^4 + o(e_n^5)$$

Proof: Suppose α be a simple root of function H such that H(α) = 0 and e_n = q_n − α where e_n be the error at q_n. We have H(q_n) and H′(q_n) by expanding Using Taylor series about ′α ′ because H is sufficiently differentiable.

$$H(q_n) = H'(\alpha)\left(e_n + C_2e_n^2 + C_3e_n^3 + C_3e_{n+o}^4(e_n^5)\right), \ldots \ldots \ldots (3)$$

And,

$$H'(q_n) = H'(\alpha)(1 + 2C_2e_n + 3C_3e_n^2 + 4C_4e_n^3 + \cdots 2\,\cdot) \ldots \ldots \ldots (4)$$

$Where,\ C_n = \frac{1}{n}\frac{H^{(n)}(\alpha)}{H^1(\alpha)}, n = 0,1,2,3 \ldots$

By dividing (3) and (4), we have

$$\frac{H(q_n)}{H'(q_n)} = e_n - C_2e_n^2 + 2(C_2^2 - C_3)e_n^3 + 0(e_n^4)$$

Now for $w_n - \alpha = e_n - \frac{H(q_n)}{H'(qn)}$, we have

$$w_n = \alpha + c_2e_n^2 + 2(c_2^2 + c_3)e_n^3 + o(e_n^4) \ldots \ldots \ldots (5)$$

Expanding H(w_n) with the Taylor series, we have

$$H(w_n) = H'(\alpha)[w_n - \alpha + C_2(w_n - \alpha)] + o((w_n - \alpha)^3)H'(\alpha)$$
$$= [C_2e_n^2 + (2C_3 - 2C_2^2)e_n^3 + (5C_2^2 - 7C_3C_2 + 3C_4)e_n^4] + o(e_n^5) \ldots \ldots \ldots (6)$$

Using (3), (4) and (6), one can get

$$H^2(q_n) - 2H^2(w_n) + Hw_nH(q_n) = e_n^2 + 3C_2e_n^3 + (-2C_2^2 + 4C_3)e_n^4 + 0(e_n^5) \ldots \ldots \ldots (7)$$

And by using (3) and (4), one can obtain,

$$H(q_n)H'(q_n) = e_n + 3C_2e_n^2 + (2C_2^2 + 4C_3)e_n^3 + 5(C_2C_3 + C_4)e_n^4 + 0(e_n^5) \ldots \ldots \ldots (8)$$

In the view of equation (7), and (8), we have the error equation of (2),

$$\frac{H^2(q_n) - 2H^2(w_n) + H(w_n)H(q_n)}{H(q_n)H'(q_n)} = e_n - 4C_2^2e_n^3 + (23C_2^3 - 15C_2C_3)e_n^4 + 0(e_n^5) \ldots \ldots \ldots (9)$$

Now using q_n and (9), we have the following final error equation.

$$q_{n+1} = q_n - \frac{H^2(q_n) - 2H^2(w_n) + H(w_n)H(q_n)}{H(q_n)H'(q_n)} = q_n - (e_n - 4C_2^2e_n^3 + (23c_2^3 - 15c_2c_3)e_n^4) \ldots \ldots \ldots (10)$$

On subtracting α from both sides of equation (3) and using $q_{n+1} - \alpha = e_{n+1}$, we get

$$e_{n+1} = (5C_2^3 - C_2C_3)e_n^4 + 0(e_n^5) \ldots \ldots \ldots (11)$$

3. Numerical Examples

This section, some real-life numerical example i.e., Continuous stirred tank reactor and academic issues are used to check the effectiveness of the suggested approach. The results of the evaluation of the suggested method's effectiveness are shown in Tables 1-2. All computations were performed using Mathematica software version 11.1.1, with a halting condition of

$$|q_{n+1} - q_n| < e,$$

Where, $e = 10^{-300}$ was used. Furthermore, this formula was used to estimate the convergence order in computation (ACOC):

$$p \approx \frac{ln\left|\frac{q_{n+2} - q_{n+1}}{q_{n+1} - q_n}\right|}{ln\left|\frac{q_{n+1} - q_n}{q_n - q_n - 1}\right|}.$$

The notation m(±n) represents m $\times 10^{(\pm n)}$, which appears throughout the table. We have used the fourth-order approach, known as SM and suggested by Soleymani [14], to allow for a meaningful comparison. It is defined as follows:

$$w_n = q_n - \frac{H(q_n)}{H'(q_n)}, q_{n+1}$$
$$= w_n - \frac{H(q_n)^2}{H(q_n)^2 - 2H(q_n)H(wn)} \frac{H(w_n)}{H'(q_n)} \left(1 + \frac{H(w_n)^2}{H(q_n)^2}\right)\left(\frac{1 + H(w_n)^2}{H'(q_n)^2}\right)\left(1 + \frac{H(q_n)^2}{H'(q_n)^2}\right) \ldots \ldots (12)$$

Furthermore, we have utilized the Chun approach (CM) [13] for the fourth-order method.

$$w_n = q_n - \frac{2}{3}\frac{H(q_n)}{H'(q_n)}, q_{n+1} = q_n + \frac{H'(q_n) + 3H'(w_n)}{2H'(q_n) - 6H'(w_n)} \frac{H(q_n)}{H'(q_n)} \ldots \ldots \ldots \; (13)$$

Example 3.1 Continuous Stirred Tank Reactor

Examine a CST reactor that is isothermal. If U and Γ are the components fed into the reactor, the reactor will create the following reaction scheme (see [15]):

U + Γ → W.
W + Γ → X.
X + Γ → Y.
Y + Γ → Z.

Douglas characterized as a fundamental feedback control mechanism (see [16]). The following equation was taken into consideration for the reactor's transfer function:

$$k_c \times \frac{2.98(q + 2.25)}{q^4 + 11.50q^3 + 47.49q^2 + 83.06325q + 51.23266875} = -1,$$

where the proportional controller's gain is denoted by Kc. The worth of Kc that cause the zeros of the transfer function to have a negative real component must be chosen for the control system's stability. Let us consider that Kc = 0, then the roots of the nonlinear equation are obtained from the singularities of the open-loop transfer function:

$$H_1(\mathrm{q}) = q^4 + 11.50q^3 + 47.49q^2 + 83.06325\mathrm{q} + 51.23266875,$$

where actual root is − 0.000006167. Taking the initial guess q _0 = −1 gives the numerical calculations presented in Table 1.

Table 1 Continuous Stirred Tank Reactor

Method	Iteration	$q_n - q_{n-1}$	$H(q_n)$	ρ
SM	2	3.31(-6)	-27.48	4.00
	3	6.85(-28)	-5.69(-31)	4.00
	4	1.26(-114)	-1.05(-107)	4.00
CM	2	1.33(-8)	1.11(1)	4.00
	3	4.27(-40)	3.55(-33)	4.00
	4	4.46(-166)	3.71(-159)	4.00
PM	2	3.78(-12)	-3.14(5)	4.00
	3	1.62(-57)	-1.34(-50)	4.00
	4	5.51(-239)	-4.57(-232)	4.00

Example 3.2 Consider a non-linear equation

$$H_2(q) = sin^2(q) - q^2 + 1$$

The actual solution of the equation is about α ≈ 1.40494…, and we start with an initial guess of 3.5. This equation is difficult to solve directly because it includes both trigonometric and polynomial terms. To find the solution, we use numerical methods, which involve repeating calculations until we get a good approximation.

Choosing a good starting guess helps the method find the correct answer more quickly. These kinds of problems are common in science, engineering, and math. Methods like Newton Raphson, bisection, and secant help find solutions to such equations. The accuracy of these methods depends on the starting guess, how the function behaves, and when we decide to stop the calculations.

Table 2 $sin^2(q) - q^2 + 1$

Method	Iteration	$q_n - q_{n-1}$	$H(q_n)$	ρ
SM	2			
	3	d	d	d
	4			
CM	2	1.75(-1)	-5.00(-1)	
	3	1.01(-3)	-2.51(-3)	3.99
	4	2.41(-12)	-5.99(-12)	
PM	2	1.37(-1)	-3.81(-1)	
	3	4.51(-4)	-7.12(-3)	4.00
	4	9.71(-14)	-2.41(-13)	

4. Conclusion

We have introduced a simple and different techniques to develop an optimal fourth order method for finding the simple roots which does not requires a second order derivatives and uses only three function evaluation such as, $\left(H(q_n), H'(q_n), H(w_n)\right)$ per full iteration. We have used the Newton's methods for solving the simple roots for solving the linear equations. Numerical examples and figures illustrate the practical usefulness of these methods. As the need for the reliable and efficient method to solve complex equation grows, these techniques are recommended for solving problems in area like numerical analysis, optimization and the computational sciences.

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