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Some Open-Newton-Cotes Type Integral inequalities for Caputo Fractional Integral Operator

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Abstract
In this paper, we present a set of open-Newton-type inequalities with n=1 for differentiable convex functions using the Caputo fractional operator. For this, first we prove an integral identity using Caputo Fractional integral and derivative operator. Further, by utilising this identity we establish some error bounds for Open-Newton-Cotes formula for differentiable convex functions and bounded functions in the fractional calculus. Finally, we added some examples and show the validity of inequalities with a graph for different values of fractional parameter α.

Keywords
Caputo fractional operator, Convex functions, Open-Newton-Cotes type inequalities.
Subject Classification: 44C20, 44A20

1. Introduction
Fractional calculus has been more popular and significant over the past three decades. It applies the ideas of integrals and derivatives to arbitrary real or complex orders. Its proved applicability in a wide range of scientific and technical areas are the reason for this increased interest. Especially in the study of special functions and their extensions over one or more variables, fractional calculus has been used to tackle challenging issues in mathematical physics and provides strong tools for solving differential and integral equations [1–8].

The theory of measure, limits, differentiation, integration, and convex functions are all included in the branch of mathematics known as mathematical analysis. The foundation of mathematical analysis is inequality, which has grown into a vital instrument in that process until the early 20th century, when the west began to see it as a distinct branch of contemporary mathematics. Hardy, Littlewood, and Poolya's book "Inequalities"[9] was the first work in this topic. Other books (see, for example, [10], [11]) are also very helpful in this field.

In the realm of mathematical analysis study, convex functions—which are basic as positive or growing functions—have become essential. The study of convex functions in relation to mathematical inequalities—most notably, Simpson's and Newton-type inequalities—has received a lot of interest in recent years. Numerous important findings in numerical analysis are based on Simpson's second rule, which is based on the 3-point Newton-Cotes quadrature rule. Newton-type inequalities, which come from three-step quadratic kernel calculations, have been the subject of substantial research and are considered as basic tools in mathematical inequalities [12-16]. Numerous mathematicians have investigated these inequalities, advanced our knowledge of their characteristics and used in a range of mathematical and scientific domains.

Numerous researchers have created numerical integration formulas in recent years and used various methodologies to determine their error bounds. The authors employed a variety of functions, including convex functions, bounded functions, Lipschitzian functions, and others, in conjunction with mathematical inequalities to ascertain the error bounds of numerical integration formulas. For example, certain error limitations for trapezoidal and midpoint formulae of numerical integration utilizing the convex functions were obtained in [17, 18].

The convex functions in various calculi have also been used to construct certain error bounds for Newton's formula in numerical integration; these bounds may be found in [19, 20, 21, 22, 23]. Milne's formula is a crucial component of open Newton-cotes formulas, and its error limits for four times twice differentiable functions were discovered in [24]. In [25], the authors employed generic form of the convexity and created several new Maclaurin's formula type inequalities and analyzed their applicability.

Nonetheless, due to their importance, some fractional integral inequalities that are helpful in approximation theory have been developed by academics using fractional calculus. The Hermite-Hadamard, Simpson's, midway, Ostrowski's, and trapezoidal inequalities are among the inequalities that may be used to identify the boundaries of mathematical integration formulae. The Hermite-Hadamard type inequality and the trapezoidal formula constraints were developed in [26]. Fractional Ostrowski's type inequalities were established in Set [27] using differentiable convexity. Iscan and Wu [28] developed an inequality of the Hermite-Hadamard type for reciprocal convex functions and established some bounds for numerical integration using Riemann-Liouville fractional integrals (RLFIs). Sitthiwirattham et al. [29] recently used the RLFIs to find some limitations for Simpson's 3/8 formula. For other inequalities that may be solved with fractional integrals, refer to [30,31,32,33,34,35,36,37] and the cited works.

Sitthiwirattham et al. [38] established some error bounds for Open-Newton-Cotes formula with n=1 for differential convex function by using Reimann-Liouville fractional operator. The Open-Newton-Cotes formula error bounds for n=1 can be found using these error bounds or inequalities, which makes them crucial in error analysis (see [39, p.200]). Motivated by the current research of them, we define new error bounds for one of the Open-Newton-Cotes formulae in fractional calculus. For differentiable convex functions, we establish bounds using Caputo fractional operator in Section 2 and provide examples to demonstrate the validity of these new bounds in section 3."

Definition 1.1 Assume that I is an interval of real numbers. Then, a function $\mathcal{F} : \mathrm{I} \rightarrow \mathrm{R}$ is said to be convex [40], if

$$\mathcal{F}(\lambda a + (1-\lambda)b) \leq \lambda\mathcal{F}(a) + (1-\lambda)\mathcal{F}(b)$$

is valid $\forall\, a, b \in I$ and $\lambda \in [0,1]$.

In this paper, we will use the well- known Caputo Fractional operators that are given below.

Definition 1.2 Let us consider $\alpha > 0$ and $\alpha \notin \{1,2,3,\ldots.\}, n = [\alpha] + 1, f \in C^n[a,b]$. The Caputo fractional derivatives [41-44] of order α are defined as follows:

$$C^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^n(t)}{(x-t)^{\alpha-n+1}} dt\,, x > a$$

And

$$C^{\alpha}_{b^-} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^n(t)}{(t-x)^{\alpha-n+1}} dt\,, x < b$$

Where $C^n[a,b]$ denotes the space of n-times differentiable functions such that f^n are continuous on $[a,b]$ and Γ denotes the well-known Gamma function that is defined below.

If $\alpha = \mathrm{n} \in \{1,2,3, \ldots\}$ and the usual derivative of order n exists, then the Caputo fractional derivative exactly matches $f^n(t)$ to a constant multiplier of $(-1)^n$.

Definition 1.3 The integral representation of the Gamma function [42-44] is defined as

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, Re(z) > 0$$

2. Open-Newton-Cotes type Inequalities for Caputo Fractional Operator

In this section, we give some Open-Newton-Cots type inequalities through various classes of functions by using Caputo Fractional Operator. To prove these inequalities firstly, we will prove the following lemma.

Lemma 2.1 If $\mathcal{F}: [\mu,\rho] \rightarrow \mathcal{R}$ be a differentiable function on (μ,ρ) such that $\mathcal{F}' \in L_1[\mu,\rho]$. If $\mathcal{F} \in C^{n+1}[\mu,\rho]$, then the following identity for Caputo fractional operator holds,

$$\begin{aligned}
&\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right) + \mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right] - \frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C^{\alpha}_{a^+}\mathcal{F}^n(\rho) + (-1)^n C^{\alpha}_{b^-}\mathcal{F}^n(\mu)\right] \\
&= \frac{\rho-\mu}{2}\int_0^{\frac{1}{3}} \lambda^{n-\alpha}\left[\mathcal{F}^{n+1}(\lambda\rho + (1-\lambda)\mu) - \mathcal{F}^{n+1}(\lambda\mu + (1-\lambda)\rho)\right] d\lambda \\
&\quad + \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\lambda^{n-\alpha} - \frac{1}{2}\right)\left[\mathcal{F}^{n+1}(\lambda\rho + (1-\lambda)\mu) - \mathcal{F}^{n+1}(\lambda\mu + (1-\lambda)\rho)\right] d\lambda \\
&\quad + \int_{\frac{2}{3}}^{1} (\lambda^{n-\alpha} - 1)\left[\mathcal{F}^{n+1}(\lambda\rho + (1-\lambda)\mu) - \mathcal{F}^{n+1}(\lambda\mu + (1-\lambda)\rho)\right] d\lambda \qquad (1)
\end{aligned}$$

Proof: The right-hand side of above aquation gives

$$=\frac{\rho-\mu}{2}\int_0^{\frac{1}{3}}\lambda^{n-\alpha}[\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)-\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)]\,d\lambda$$
$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\lambda^{n-\alpha}-\frac{1}{2}\right)[\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)-\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)]\,d\lambda$$
$$+\int_{\frac{2}{3}}^{1}(\lambda^{n-\alpha}-1)[\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)-\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)]\,d\lambda$$
$$=\frac{\rho-\mu}{2}[I_1-I_2+I_3-I_4+I_5-I_6] \qquad (2)$$

With the help of integration by parts, we have

$$I_1=\int_0^{\frac{1}{3}}\lambda^{n-\alpha}\,\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)d\lambda$$
$$=\frac{1}{\rho-\mu}\left[\lambda^{n-\alpha}\mathcal{F}^{n}|\mu+(1-\lambda)\rho|_0^{1/3}\right]-(n-\alpha)\int_0^{\frac{1}{3}}\lambda^{n-\alpha-1}[\mathcal{F}^{n}(\lambda\rho+(1-\lambda)\mu)]\,d\lambda$$
$$=\frac{1}{\rho-\mu}\left[\left(\frac{1}{3}\right)^{n-\alpha}\mathcal{F}^{n}\left(\frac{2\mu+\rho}{3}\right)-(n-\alpha)\int_0^{\frac{1}{3}}\lambda^{n-\alpha-1}[\mathcal{F}^{n}(\lambda\rho+(1-\lambda)\mu)]\,d\lambda\right]$$
$$I_3=\int_0^{\frac{1}{3}}\left(\lambda^{n-\alpha}-\frac{1}{2}\right)\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)d\lambda$$
$$=\frac{1}{\rho-\mu}\left[\left(\left(\frac{2}{3}\right)^{n-\alpha}-\frac{1}{2}\right)-\mathcal{F}^{n}\left(\frac{\mu+2\rho}{3}\right)-\left(\left(\frac{1}{3}\right)^{n-\alpha}-\frac{1}{2}\right)\mathcal{F}^{n}\left(\frac{2\mu+\rho}{3}\right)-(n-\alpha)\int_{\frac{1}{3}}^{\frac{2}{3}}\lambda^{n-\alpha-1}[\mathcal{F}^{n}(\lambda\rho+(1-\lambda)\mu)]\,d\lambda\right]$$
$$I_5=\int_{\frac{2}{3}}^{1}(\lambda^{n-\alpha}-1)[\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)]\,d\lambda$$
$$=\frac{1}{\rho-\mu}\left[\left(1-\left(\frac{2}{3}\right)^{n-\alpha}\right)\mathcal{F}^{n}\left(\frac{\mu+2\rho}{3}\right)-(n-\alpha)\int_{\frac{2}{3}}^{1}\lambda^{n-\alpha-1}[\mathcal{F}^{n}(\lambda\rho+(1-\lambda)\mu)]\,d\lambda\right]$$

Then using definition 1.2, we have

$$\frac{(\rho-\mu)}{2}[I_1+I_3+I_5]=\frac{1}{4}\left[\mathcal{F}^{n}\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^{n}\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{(\rho-\mu)^{n-\alpha}}(-1)^n C_{b^-}^{\alpha}\mathcal{F}^{n}(\mu) \quad (3)$$

in the same manner, we have

$$\frac{(\rho-\mu)}{2}[I_2+I_4+I_6]=-\frac{1}{4}\left[\mathcal{F}^{n}\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^{n}\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{(\rho-\mu)^{n-\alpha}}C_{a^+}^{\alpha}\mathcal{F}^{n}(\rho) \quad (4)$$

By plugging equation 3 and equation 4 in equation 2, we get the required identity.

Theorem 2.2 Let f satisfies the assumption of lemma 2.1 and the function $|\mathcal{F}^{n+1}|$ is convex on the interval $[\mu,\rho]$. Then the following inequality holds:

$$\left|\frac{1}{2}\left[\mathcal{F}^{n}\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^{n}\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^{n}(\rho)+(-1)^n C_{b^-}^{\alpha}\mathcal{F}^{n}(\mu)\right]\right|$$
$$\leq\frac{(\rho-\mu)}{2}\left[\frac{1+2^{n-\alpha+1}+(n-\alpha-2)3^{n-\alpha}}{3^{n-\alpha+1}(n-\alpha+1)}+\mathcal{A}_1(n,\alpha)\right]\left[\left|\mathcal{F}^{n+1}(\mu)+|\mathcal{F}^{n+1}(\rho)|\right|\right]$$

with $\alpha>0, n=[\alpha]+1$ and Where,

$$\mathcal{A}_1(n,\alpha)=\begin{cases}\dfrac{2^{n-\alpha+1}-1}{3^{n-\alpha+1}(n-\alpha+1)}-\dfrac{1}{6}, & 0<\alpha\leq\dfrac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{3}\right)}\\ \left(\dfrac{1}{2}\right)^{\frac{1}{n-\alpha}}+\dfrac{2^{n-\alpha+1}+1}{3^{n-\alpha+1}(n-\alpha+1)}-2\dfrac{\left(\frac{1}{2}\right)^{\frac{n-\alpha+1}{n-\alpha}}}{n-\alpha+1}-\dfrac{1}{2}, & \dfrac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{1}{3}\right)}<\alpha\leq\dfrac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}\\ \dfrac{1}{6}-\dfrac{2^{n-\alpha+1}-1}{3^{n-\alpha+1}(n-\alpha+1)}, & \dfrac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{2}{3}\right)}<\alpha\leq 1\end{cases}$$

Proof: Taking modulus in equation 1, we have

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho)+(-1)^nC_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|$$
$$=\frac{\rho-\mu}{2}\int_0^{\frac{1}{3}}\lambda^{n-\alpha}\left[|\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)|-|\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)|\right]d\lambda$$
$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|\left[|\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)|-|\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)|\right]d\lambda$$
$$+\int_{\frac{2}{3}}^{1}(\lambda^{n-\alpha}-1)\left[|\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)|-|\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)|\right]d\lambda$$

using the convexity of $|\mathcal{F}^{n+1}|$, one can obtain

$$\leq\frac{(\rho-\mu)}{2}\left[|\mathcal{F}^{n+1}(\mu)+|\mathcal{F}^{n+1}(\rho)||\right]\left[\int_0^{\frac{1}{3}}\lambda^{n-\alpha}d\lambda+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|d\lambda+\int_{\frac{2}{3}}^{1}(\lambda^{n-\alpha}-1)\,d\lambda\right]$$
$$=\frac{(\rho-\mu)}{2}\left[\frac{1}{3^{n-\alpha+1}(n-\alpha+1)}+\mathcal{A}_1(n,\alpha)+\frac{2^{n-\alpha+1}+(n-\alpha-2)3^{n-\alpha}}{3^{n-\alpha+1}(n-\alpha+1)}\right]\left[|\mathcal{F}^{n+1}(\mu)+|\mathcal{F}^{n+1}(\rho)||\right]$$

Thus, the proof is completed.

Remark: When we set $\alpha=0, n=1$ then we have the following inequality:

$$\left|\frac{1}{2}\left[\mathcal{F}\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{1}{2(\rho-\mu)}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho)+(-1)^nC_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|\leq\frac{5(\rho-\mu)}{72}\left[\mathcal{F}'|(\mu)|+|\mathcal{F}'(\rho)||\right]$$

Theorem 2.3 Consider that the assumptions in lemma 2.1 and the function $|\mathcal{F}^{n+1}|^q$, $q>1$ is convex on $[\mu,\rho]$. Then, the following inequality holds:

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho)+(-1)^nC_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|$$
$$\leq(\rho-\mu)\left[\left(\frac{(1/3)^{p(n-\alpha)+1}}{p(n-\alpha)+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+5|\mathcal{F}^{n+1}(\mu)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}^{n+1}(\mu)|^q+5|\mathcal{F}^{n+1}(\rho)|^q}{18}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|^p d\lambda\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+|\mathcal{F}^{n+1}(\mu)|^q}{6}\right)^{\frac{1}{q}}\right]$$

with $\alpha>0, n=[\alpha]+1$ $n=[\alpha]+1$ and Where, $p+q=pq$.

Proof: Taking modulus of inequality (1) and using Holder inequality, we have

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho)+(-1)^nC_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|$$

$$\leq \frac{(\rho-\mu)}{2}\left[\left(\int_0^{\frac{1}{3}}|\lambda^{n-\alpha}|^p d\lambda\right)^{\frac{1}{p}}\left(\left(\int_0^{\frac{1}{3}}|\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)|^q d\lambda\right)^{\frac{1}{q}}+\left(\int_0^{\frac{1}{3}}|\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)|^q d\lambda\right)^{\frac{1}{q}}\right)\right.$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|^p d\lambda\right)^{\frac{1}{p}}\left(\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)|^q d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}|\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)|^q d\lambda\right)^{\frac{1}{q}}\right)$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}|1-\lambda^{n-\alpha}|^p d\lambda\right)^{\frac{1}{p}}\left(\left(\int_{\frac{2}{3}}^{1}|\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)|^q d\lambda\right)^{\frac{1}{q}}+\left(\int_{\frac{2}{3}}^{1}|\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)|^q d\lambda\right)^{\frac{1}{q}}\right)\right]$$

We have the following relation by using the convexity of $|\mathcal{F}^{n+1}|^q, q>1$

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho)+(-1)^n C_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|$$

$$\leq \frac{(\rho-\mu)}{2}\left[\left(\int_0^{\frac{1}{3}}|\lambda^{n-\alpha}|^p d\lambda\right)^{\frac{1}{p}}\left(\left(\int_0^{\frac{1}{3}}\lambda|\mathcal{F}^{n+1}(\rho)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\mu)|^q d\lambda\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\int_0^{\frac{1}{3}}\lambda|\mathcal{F}^{n+1}(\mu)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\rho)|^q d\lambda\right)^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|^p d\lambda\right)^{\frac{1}{p}}\left(\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\lambda|\mathcal{F}^{n+1}(\rho)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\mu)|^q d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\lambda|\mathcal{F}^{n+1}(\mu)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\rho)|^q\right)^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{2}{3}}^{1}|1-\lambda^{n-\alpha}|^p d\lambda\right)^{\frac{1}{p}}\left(\left(\int_{\frac{2}{3}}^{1}\lambda|\mathcal{F}^{n+1}(\rho)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\mu)|^q d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_{\frac{2}{3}}^{1}\lambda|\mathcal{F}^{n+1}(\mu)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\rho)|^q\right)^{\frac{1}{q}}\right)\right]$$

$$= \frac{(\rho-\mu)}{2}\left[\left(\frac{(1/3)^{p(n-\alpha)+1}}{p(n-\alpha)+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+5|\mathcal{F}^{n+1}(\mu)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}^{n+1}(\mu)|^q+5|\mathcal{F}^{n+1}(\rho)|^q}{18}\right)^{\frac{1}{q}}\right)\right.$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|^p d\lambda\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+|\mathcal{F}^{n+1}(\mu)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}^{n+1}(\mu)|^q+5|\mathcal{F}^{n+1}(\rho)|^q}{6}\right)^{\frac{1}{q}}\right)(\rho$$

$$-\mu)\left[\left(\frac{(1/3)^{p(n-\alpha)+1}}{p(n-\alpha)+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+5|\mathcal{F}^{n+1}(\mu)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}^{n+1}(\mu)|^q+5|\mathcal{F}^{n+1}(\rho)|^q}{18}\right)^{\frac{1}{q}}\right)\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|^p d\lambda\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+|\mathcal{F}^{n+1}(\mu)|^q}{6}\right)^{\frac{1}{q}}\right]$$

$$\left.+\left(\frac{(1/3)^{p(n-\alpha)+1}}{p(n-\alpha)+1)}\right)^{\frac{1}{p}}\left(\left(\frac{5|\mathcal{F}^{n+1}(\rho)|^q+|\mathcal{F}^{n+1}(\mu)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{5|\mathcal{F}^{n+1}(\mu)|^q+|\mathcal{F}^{n+1}(\rho)|^q}{18}\right)^{\frac{1}{q}}\right)\right]$$

$$= (\rho-\mu)\left[\left(\frac{(1/3)^{p(n-\alpha)+1}}{p(n-\alpha)+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+5|\mathcal{F}^{n+1}(\mu)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}^{n+1}(\mu)|^q+5|\mathcal{F}^{n+1}(\rho)|^q}{18}\right)^{\frac{1}{q}}\right)\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|^p d\lambda\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+|\mathcal{F}^{n+1}(\mu)|^q}{6}\right)^{\frac{1}{q}}\right]$$

Thus, the proof is completed.

Remark: When we set $\alpha = 1$, then we have the following inequality

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{1}{2(\rho-\mu)}\int_\mu^\rho \mathcal{F}^n(\lambda)d\lambda\right|$$

$$\leq (\rho-\mu)\left[\left(\frac{(1/3)^{p(n-1)+1}}{p(n-1)+1)}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+5|\mathcal{F}^{n+1}(\mu)|^q}{18}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{F}^{n+1}(\mu)|^q+5|\mathcal{F}^{n+1}(\rho)|^q}{18}\right)^{\frac{1}{q}}\right)\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-1}-\frac{1}{2}\right|^p d\lambda\right)^{\frac{1}{p}}\left(\frac{|\mathcal{F}^{n+1}(\rho)|^q+|\mathcal{F}^{n+1}(\mu)|^q}{6}\right)^{\frac{1}{q}}\right]$$

Theorem 2.4 Consider that the assumptions in lemma 2.1 and the function $|\mathcal{F}^{n+1}|^q$, $q > 1$ is convex on $[\mu,\rho]$. Then, the following inequality holds:

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho)+(-1)^n C_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|$$

$$\leq \frac{(\rho-\mu)}{2}\left[\left(\frac{(1/3)^{n-\alpha+1}}{n-\alpha+1}\right)^{1-\frac{1}{q}}\left([\varphi_1(\alpha)|\mathcal{F}^{n+1}(\rho)|^q+\varphi_2(\alpha)|\mathcal{F}^{n+1}(\mu)|^q]^{\frac{1}{q}}\right.\right.$$

$$\left.+[\varphi_1(\alpha)|\mathcal{F}^{n+1}(\mu)|^q+\varphi_2(\alpha)|\mathcal{F}^{n+1}(\rho)|^q]^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|d\lambda\right)^{1-\frac{1}{q}}\left([\varphi_3(\alpha)|\mathcal{F}^{n+1}(\rho)|^q+\varphi_4(\alpha)|\mathcal{F}^{n+1}(\mu)|^q]^{\frac{1}{q}}\right.$$

$$\left.+[\varphi_3(\alpha)|\mathcal{F}^{n+1}(\mu)|^q+\varphi_4(\alpha)|\mathcal{F}^{n+1}(\rho)|^q]^{\frac{1}{q}}\right)$$

$$+\left(\frac{(1/3)^{n-\alpha+1}}{n-\alpha+1}\right)^{1-\frac{1}{q}}\left([\varphi_5(\alpha)|\mathcal{F}^{n+1}(\rho)|^q+\varphi_6(\alpha)|\mathcal{F}^{n+1}(\mu)|^q]^{\frac{1}{q}}\right.$$

$$\left.\left.+[\varphi_5(\alpha)|\mathcal{F}^{n+1}(\mu)|^q+\varphi_6(\alpha)|\mathcal{F}^{n+1}(\rho)|^q]^{\frac{1}{q}}\right)\right]$$

with $\alpha>0, n=[\alpha]+1$ and Where, $p+q=pq$.

$$\varphi_1(\alpha)=\int_0^{\frac{1}{3}}\lambda|\lambda^{n-\alpha}|d\lambda$$

$$\varphi_2(\alpha)=\int_0^{\frac{1}{3}}(1-\lambda)|\lambda^{n-\alpha}|d\lambda$$

$$\varphi_3(\alpha)=\int_{\frac{1}{3}}^{\frac{2}{3}}\lambda\left|\lambda^{n-\alpha}-\frac{1}{2}\right|d\lambda$$

$$\varphi_4(\alpha)=\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\lambda)\left|\lambda^{n-\alpha}-\frac{1}{2}\right|d\lambda$$

$$\varphi_5(\alpha)=\int_{\frac{2}{3}}^{1}\lambda(1-\lambda^{n-\alpha})\,d\lambda$$

$$\varphi_6(\alpha)=\int_{\frac{2}{3}}^{1}(1-\lambda)(1-\lambda^{n-\alpha})\,d\lambda$$

Proof: Taking modulus of inequality (1) and using power-mean inequality, we have

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho)+(-1)^n C_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|$$

$$\leq \frac{(\rho-\mu)}{2}\left[\left(\int_0^{\frac{1}{3}}|\lambda^{n-\alpha}|d\lambda\right)^{1-\frac{1}{q}}\left(\left(\int_0^{\frac{1}{3}}|\lambda^{n-\alpha}||\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)|^q d\lambda\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\int_0^{\frac{1}{3}}|\lambda^{n-\alpha}||\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)|^q d\lambda\right)^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|d\lambda\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right||\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)|^q d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right||\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)|^q d\lambda\right)^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{2}{3}}^{1}|1-\lambda^{n-\alpha}|d\lambda\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{2}{3}}^{1}|1-\lambda^{n-\alpha}||\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)|^q d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_{\frac{2}{3}}^{1}|1-\lambda^{n-\alpha}||\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)|^q d\lambda\right)^{\frac{1}{q}}\right)\right]$$

We have the following relation by using the convexity of $|\mathcal{F}^{n+1}|^q, q>1$

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho)+(-1)^n C_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|$$

$$\leq \frac{(\rho-\mu)}{2}\left[\left(\frac{(1/3)^{n-\alpha+1}}{n-\alpha+1}\right)^{1-\frac{1}{q}}\left(\left(\int_0^{\frac{1}{3}}|\lambda^{n-\alpha}|[\lambda|\mathcal{F}^{n+1}(\rho)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\mu)|^q]d\lambda\right)^{\frac{1}{q}}\right.\right.$$

$$\left.+\left(\int_0^{\frac{1}{3}}|\lambda^{n-\alpha}|[\lambda|\mathcal{F}^{n+1}(\mu)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\rho)|^q]d\lambda\right)^{\frac{1}{q}}\right)$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|d\lambda\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|[\lambda|\mathcal{F}^{n+1}(\rho)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\mu)|^q]d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|[\lambda|\mathcal{F}^{n+1}(\mu)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\rho)|^q]d\lambda\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{(1/3)^{n-\alpha+1}}{n-\alpha+1}\right)^{1-\frac{1}{q}}\left(\left(\int_{\frac{2}{3}}^{1}|1-\lambda^{n-\alpha}|[|\mathcal{F}^{n+1}(\rho)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\mu)|^q]d\lambda\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_{\frac{2}{3}}^{1}|1-\lambda^{n-\alpha}|[\lambda|\mathcal{F}^{n+1}(\mu)|^q+(1-\lambda)|\mathcal{F}^{n+1}(\rho)|^q]d\lambda\right)^{\frac{1}{q}}\right)\right]$$

Thus, the proof is completed.

Theorem 2.5 Suppose that the conditions of Proposition 1 are valid. Then, there exist $m, M \in R$ such that $m \leq \mathcal{F}^{n+1}(\lambda) \leq M$ for $\lambda \in [\mu,\rho]$. with $\alpha > 0, n = [\alpha]+1$ Under these conditions, the following Newton-type inequality holds:

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C^{\alpha}_{a^+}\mathcal{F}^n(\rho)+(-1)^n C^{\alpha}_{b^-}\mathcal{F}^n(\mu)\right]\right|$$
$$\leq \frac{\rho-\mu}{2}(M-m)\left\{\int_0^{\frac{1}{3}}|\lambda^{n-\alpha}|d\lambda+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\lambda^{n-\alpha}-\frac{1}{2}\right|d\lambda+\int_{\frac{2}{3}}^{1}|1-\lambda^{n-\alpha}|d\lambda\right\}$$

Proof: By using the lemma 2.1, we have

$$\left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu+\rho}{3}\right)+\mathcal{F}^n\left(\frac{\mu+2\rho}{3}\right)\right]-\frac{\Gamma(n-\alpha+1)}{2(\rho-\mu)^{n-\alpha}}\left[C^{\alpha}_{a^+}\mathcal{F}^n(\rho)+(-1)^n C^{\alpha}_{b^-}\mathcal{F}^n(\mu)\right]\right|$$

$$=\frac{\rho-\mu}{2}\int_0^{\frac{1}{3}}\lambda^{n-\alpha}([\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)-m+M]+[m+M-\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)])\,d\lambda$$

$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\lambda^{n-\alpha}-\frac{1}{2}\right)([\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)-m+M]+[m+M-\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)])\,d\lambda$$

$$+\int_{\frac{2}{3}}^{1}(\lambda^{n-\alpha}-1)([\mathcal{F}^{n+1}(\lambda\rho+(1-\lambda)\mu)-m+M]+[m+M\mathcal{F}^{n+1}(\lambda\mu+(1-\lambda)\rho)])\,d\lambda$$

If the absolute value of above equation is taken, then

$$= \frac{\rho - \mu}{2} \int_0^{\frac{1}{3}} \lambda^{n-\alpha} (|\mathcal{F}^{n+1}(\lambda\rho + (1-\lambda)\mu) - m + M| + |m + M - \mathcal{F}^{n+1}(\lambda\mu + (1-\lambda)\rho)|)\, d\lambda$$
$$+ \int_{\frac{1}{3}}^{\frac{2}{3}} \left|\lambda^{n-\alpha} - \frac{1}{2}\right| (|\mathcal{F}^{n+1}(\lambda\rho + (1-\lambda)\mu) - m + M| + |m + M - \mathcal{F}^{n+1}(\lambda\mu + (1-\lambda)\rho)|) d\lambda$$
$$+ \int_{\frac{2}{3}}^{1} |\lambda^{n-\alpha} - 1| (|\mathcal{F}^{n+1}(\lambda\rho + (1-\lambda)\mu) - m + M| + |m + M - \mathcal{F}^{n+1}(\lambda\mu + (1-\lambda)\rho)|)\, d\lambda$$

It is known that $m \leq \mathcal{F}^{n+1}(\lambda) \leq M$ for $\lambda \in [\mu, \rho]$. Then it follows

$$|\mathcal{F}^{n+1}(\lambda\rho + (1-\lambda)\mu) - m + M| \leq M - m$$
$$|m + M - \mathcal{F}^{n+1}(\lambda\mu + (1-\lambda)\rho)| \leq M - m$$

With the help of this, we get

$$\leq \frac{\rho - \mu}{2}(M - m)\left\{\int_0^{\frac{1}{3}} |\lambda^{n-\alpha}| d\lambda + \int_{\frac{1}{3}}^{\frac{2}{3}} \left|\lambda^{n-\alpha} - \frac{1}{2}\right| d\lambda + \int_{\frac{2}{3}}^{1} |1 - \lambda^{n-\alpha}| d\lambda\right\}$$

Thus, the proof is completed.

3. Examples

In this section, we will provide some mathematical examples and their graphs to show the validity of two new inequalities.

Example 3.1 Let $f: [0,1] \to \mathbb{R}$ be a function such that $f(x) = x^3$ and $f''(x) = 6x$ is a convex function, then for $\alpha \in (0.01, 0.99)$ and $n = 1$, from theorem 2.2

$$LHS = \left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu + \rho}{3}\right) + \mathcal{F}^n\left(\frac{\mu + 2\rho}{3}\right)\right] - \frac{\Gamma(n - \alpha + 1)}{2(\rho - \mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho) + (-1)^n C_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|$$
$$= \left|\frac{5}{6} - \frac{3}{2(2-\alpha)}\right|$$
$$RHS = 3\left[\frac{2.2^{2-\alpha} - (1+\alpha).3^{1-\alpha}}{3^{2-\alpha}(2-\alpha)} - \frac{1}{6}\right]$$

From the figure 1, it is clear that $LHS \leq RHS$.

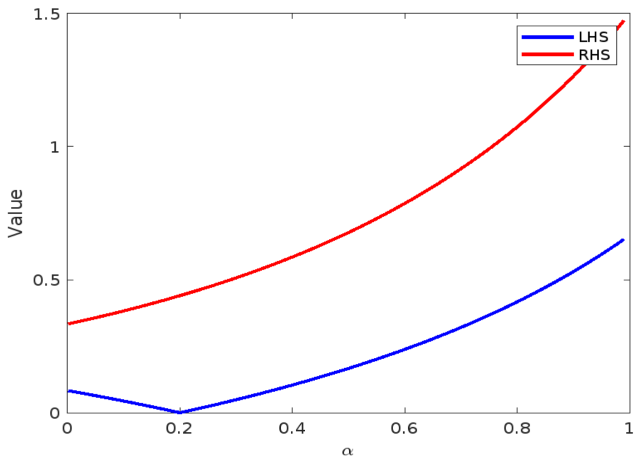

Fig. 1 An example to the inequality 2.2

Example 3.2 Let $f: [0,1] \to \mathbb{R}$ be a function such that $f(x) = x^3$ and $|f''(x)|^q = 36x^2$ is a convex function on [0,1]. Therefore we apply Theorem 2.4 to this defined function f for $\alpha \in (0.01, 0.99)$ and $n = 1$. The left hand side of the inequality from theorem 2.4 is

$$LHS = \left|\frac{1}{2}\left[\mathcal{F}^n\left(\frac{2\mu + \rho}{3}\right) + \mathcal{F}^n\left(\frac{\mu + 2\rho}{3}\right)\right] - \frac{\Gamma(n - \alpha + 1)}{2(\rho - \mu)^{n-\alpha}}\left[C_{a^+}^{\alpha}\mathcal{F}^n(\rho) + (-1)^n C_{b^-}^{\alpha}\mathcal{F}^n(\mu)\right]\right|$$

$$= \left|\frac{5}{6} - \frac{3}{2(2-\alpha)}\right|$$

$$RHS = \left(\frac{1}{3^{(3-2\alpha}((3-2\alpha)}\right)^{\frac{1}{2}} \left(\sqrt{2} + \sqrt{10}\right) + \sqrt{6}\left(\frac{2^{3-\alpha}-1}{3^{3-2\alpha}(3-2\alpha)} - \frac{2^{2-\alpha}-1}{3^{2-\alpha}(2-\alpha)} + \frac{1}{12}\right)^{\frac{1}{2}}$$

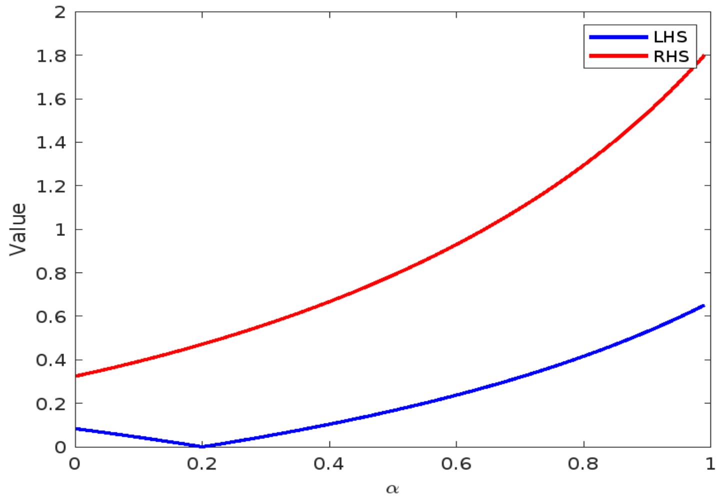

Fig. 2 An example to the inequality 2.2

From the figure 2, it is clear that $LHS \leq RHS$.

4. Conclusion

In summary, the present article introduces a number of Open-Newton-Cotes type inequalities derived using different classes of functions and the Caputo fractional operator. The study starts with an integral identity that is essential for proving the main results, followed by a variety of Open-Newton-Cotes type inequalities for differentiable convex functions using fractional integrals. In addition, inequalities are developed for bounded functions within this framework, and we provided some examples and their graphs to demonstrate the validity of the newly established inequalities for various values of α. The methods and approaches employed to connect the Caputo fractional operator with Newton-type inequalities could open up new research directions, such as exploring different classes of functions or applying other fractional integral operators.

Conflict of Interest

There is no conflict of interest to declare.

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